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Robust Stability: The Computational Complexity Point of View

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In this paper, we explore the new and emerging research area of *robust stability* and study its interplay with computational complexity. Robust stability deals with a *family* $\mathcal{P}$ consisting of all polynomials $p(s, q)$ of fixed order n whose coefficients vary in a set $Q \subset \mathbf{R}^{n+1}$. The main task of robust stability is to detect if all the roots of $p(s, q)$ are contained in a given region $\mathcal{D}$ of the complex plane for all $q \in Q$. In the special case when $\mathcal{D}$ is the open left half plane and $\mathcal{P}$ is a so-called *interval polynomial* we combine the *Theorem of Kharitonov* with the *Test of Routh* and show that the number of elementary operations (multiplications/divisions and additions/subtractions) required for the solution of this problem is at most $O(n^2)$. © 1992 Academic Press, Inc.

I. INTRODUCTION

Quite recently, the efforts of many researchers operating in the area of robust control have been devoted to stability properties of uncertain systems. The striking result of Kharitonov, dealing with the so-called Hurwitz property of interval polynomials, provided the motivations for

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studying *extreme point results* for robustness analysis and design of control systems. Roughly speaking, an extreme point result means that a set of "specially constructed" polynomials enjoy the same property (for example, robust stability) as the entire family. A complete list of extreme point results is reported in the work of Barmish and Kang (to appear). Additionally, the survey papers of Barmish (1988) and Jury (1990) provide a good overview of this new research area which has been largely motivated by the breakthrough of Kharitonov.

In robust stability, an interesting, but apparently neglected, line of research is to evaluate the number of elementary operations needed to solve a specific problem; see the work of Tempo (to appear). The motivation for this research is obvious: When a certain stability test is implemented on a digital computer it is crucial to know the minimal number of elementary operations required.

The main goal of this paper is to provide a first attempt to merge the areas of robust stability and computational complexity. To this end, we first overview some extreme point results, and then we evaluate the computational complexity of some specific problems of interest in many applications in systems and control. We remark that it is not our intention to provide a detailed list of results in the robustness area since the interested reader can refer to the survey papers cited above. Here, we report only *some* extreme point results—the ones we deem more appealing for a combination of robust stability and computational complexity. For example, we only include extreme point results for polynomial families and not for families of transfer functions. For completeness and because of their major impact on robust stability, we also add a few results which are not in the spirit of extreme point results, for example, the so-called *Edge Theorem*.

II. Preliminaries

A *family* of real polynomials $\mathcal{P}_p$ is described by

$$\mathcal{P}_p = \left\{ p(s, q) : p(s, q) = p^*(s) + \sum_{i=0}^{n} q_i s^i,\ q \in Q_p \right\}, \tag{1}$$

where $p^*(s)$ is a fixed nth order polynomial

$$p^*(s) \doteq \sum_{i=0}^{n} q_i^* s^i \tag{2}$$

and Q_p is the l_p ball of fixed radius $r > 0$

$$Q_p \doteq \{q : \|q\|_p \leq r\}. \tag{3}$$

In this work, we mainly restrict our attention to the special cases when $p = 1, 2, \infty$ and include some brief discussions on other families of polynomials. Throughout the paper we assume that all polynomials in $\mathcal{P}_p$ have degree equal to n. Note that this "no degree dropping" requirement is automatically satisfied if $p(s, q)$ is monic. In the case when $p(s, q)$ is not a monic polynomial, this assumption is equivalent to

$$|q_n^*| > r. \tag{4}$$

Formally, we say that $\mathcal{P}_p$ is *robustly stable* if all the roots of its members $p(s, q)$ are contained in a given region $\mathcal{D}$ of the complex plane. Similarly, in the special case when $\mathcal{D}$ is the open left half plane (unit disk), the family $\mathcal{P}_p$ is called *Hurwitz* (*Schur*). In this framework, it is also of interest to compute the largest value $r_{\max}$ of the radius r such that the family $\mathcal{P}_p$ is robustly stable. The value $r_{\max}$ is often called the *robustness margin*.

The *computational complexity* of robust stability is defined as the minimal number of elementary operations (multiplications, divisions, additions, and subtractions) needed to check if $\mathcal{P}_p$ is robustly stable. In the following, any upper bound of the computational complexity is denoted by COMP.

III. Some Results for Hurwitz Stability and l_p Norms

The Theorem of Kharitonov for l_∞ Norms

In this subsection, we study the case $p = \infty$. That is, the coefficients q of $p(s, q)$ vary in the $(n + 1)$-dimensional rectangle

$$Q_\infty \doteq \{q \in R^{n+1} : q_i^- \leq q_i \leq q_i^+, \ i = 0, 1, \ldots, n\}, \tag{5}$$

where $q_i^- = q_i^* - r$ and $q_i^+ = q_i^* + r$ for $i = 0, 1, \ldots, n$. We say that $p(s, q^v)$ is a *vertex polynomial* associated with the vertex q^v of the rectangle Q_∞. Since the coefficients of $p(s, q)$ vary independently between upper and lower bounds, the family $\mathcal{P}_\infty$ is called an *interval polynomial*. The Theorem of Kharitonov (1978) shows that Hurwitzness of the family $\mathcal{P}_\infty$ is equivalent to Hurwitzness of *four* vertex polynomials, the so-called *Kharitonov polynomials*.

THEOREM 1. *The family $\mathcal{P}_\infty$ defined in* (1) *for $p = \infty$ is Hurwitz if and only if the four Kharitonov polynomials*

$$\begin{aligned}
p_1(s) &\doteq q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \cdots, \\
p_2(s) &\doteq q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \cdots, \\
p_3(s) &\doteq q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \cdots, \\
p_4(s) &\doteq q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + \cdots,
\end{aligned} \tag{6}$$

are Hurwitz.

For the interested reader, we mention that the original work of V. L. Kharitonov was rather cryptic. In addition, a crucial step in the proof was the Theorem of Hermite–Bieler on the so-called "interlacing property of Hurwitz polynomials" (see Vol. 2 of Gantmacher, 1960). More recently, Minnichelli *et al.* (1989) have shown a simpler proof which uses only two-dimensional geometric ideas and the well-known "monotonic phase property of Hurwitz polynomials" (also named principle of argument or Mikhailov criterion; e.g., see Netushil, 1973).

We remark that the Theorem of Kharitonov can be simplified for low order polynomials. In particular, Anderson *et al.* (1987) proved that for $n < 6$ only a subset of the Kharitonov polynomials are needed.

Finally, we recall that Fu and Barmish (1988) provided a closed-form formula for the robustness margin of the family $\mathcal{P}_\infty$. This formula is given in terms of the eigenvalues of a "specially constructed matrix" and is based on certain weighted polynomials having a structure similar to that of the Kharitonov polynomials.

Robust Stability for l_2 Norms

We now consider the case when the coefficients q vary in the $(n + 1)$-dimensional sphere

$$Q_2 = \left\{ q \in R^{n+1} : \sqrt{\sum_{i=0}^{n} q_i^2} \le r \right\}. \tag{7}$$

At present, there are no extreme point results for the family $\mathcal{P}_2$. However, for completeness we include a test which is not in the spirit of extreme points and involves a one-dimensional optimization problem.

For $s = j\omega$, $j = \sqrt{-1}$, let $\operatorname{Re} p(j\omega, 0)$ and $\operatorname{Im} p(j\omega, 0)$ denote the real and imaginary parts of $p(s, 0)$ and define a *frequency testing function* as

$$T(\omega) \doteq \frac{(\operatorname{Re} p(j\omega, 0))^2}{\sum_{i \text{ even}} \omega^{2i}} + \frac{(\operatorname{Im} p(j\omega, 0))^2}{\sum_{i \text{ odd}} \omega^{2i}}. \tag{8}$$

In addition, we define the minimum value of this function as

$$T^* \doteq \inf_{\omega} T(\omega). \tag{9}$$

We now report the result of Soh *et al.* (1985) which gives a formula for detecting robust stability.

THEOREM 2. *The family* $\mathcal{P}_2$ defined in (1) for $p = 2$ is Hurwitz if and only if

$$\begin{aligned} q_0^* &> r; \\ q_n^* &> r; \\ \sqrt{T^*} &> r. \end{aligned} \tag{10}$$

Using Theorem 2, it is straightforward to show that the robustness margin $r_{\max}$ is given by

$$r_{\max} = \min\{\sqrt{T^*}, |q_0^*|, |q_n^*|\}. \tag{11}$$

Robust Stability for l_1 *Norms*

Motivated by the Theorem of Kharitonov, a recent branch of the literature has concentrated on the so-called dual problem where a *diamond polynomial* family is studied; e.g., see Tempo (1988, 1990); Bose and Kim (1989); Katbab and Jury (1990); Kang *et al.* (1991); Barmish *et al.* (1992). Within this framework, the coefficients q of $p(s, q)$ lie in the $(n + 1)$-dimensional diamond

$$Q_1 = \left\{q \in R^{n+1} : \sum_{i=0}^{n} |q_i| \leq r\right\}. \tag{12}$$

For this uncertainty structure, Barmish *et al.* (1992) proved that it suffices to check the Hurwitz stability of *eight* vertex polynomials.

THEOREM 3. *The family* $\mathcal{P}_1$ *defined in* (1) *for* $p = 1$ *is Hurwitz if and only if the eight polynomials*

$$\begin{aligned} p_1(s) &\doteq p^*(s) + r; \\ p_2(s) &\doteq p^*(s) - r; \\ p_3(s) &\doteq p^*(s) + rs; \\ p_4(s) &\doteq p^*(s) - rs; \end{aligned}$$

$$p_5(s) \doteq p^*(s) + rs^{n-1};$$
$$p_6(s) \doteq p^*(s) - rs^{n-1};$$
$$p_7(s) \doteq p^*(s) + rs^{n};$$
$$p_8(s) \doteq p^*(s) - rs^{n} \tag{13}$$

are Hurwitz.

The proof of this result is mainly geometric and uses the so-called *value set* concept (see Barmish, 1988). Additionally, a Hurwitz preserving transformation between real coefficients polynomials and complex coefficients polynomials is invoked (see Chap. 3 of Jury, 1982, for details).

IV. Some Extreme Point Results for More General $\mathfrak{D}$ Regions and Interval Polynomials

Weak and Strong Kharitonov Regions

Motivated by the theory shown in the previous sections, we report some extreme point results for interval polynomials and the so-called *weak* and *strong Kharitonov regions*. Examples of weak Kharitonov regions, listed in the paper of Petersen (1989) (see also Fu, 1991), are the shifted unit disk, the left sector, the hyperbolic region, and the elliptic region; an example of a strong Kharitonov region is the "special sector" defined by Soh and Foo (1990). More formally,

—If root location of an interval polynomial in a certain $\mathfrak{D}$ region of the complex plane can be ascertained by means of *all* the vertex polynomials, then $\mathfrak{D}$ is a weak Kharitonov region.

—Similarly, if root location can be detected by means of a subset of vertex polynomials which is independent of the order of the polynomial, then $\mathfrak{D}$ is a strong Kharitonov region.

Clearly, the number of vertex polynomials N_v one has to check for the "weak problem" is

$$N_v = 2^{n+1}. \tag{14}$$

The complete description of a weak Kharitonov region is given by the following result of Rantzer (to appear) for the case of complex polynomials:

THEOREM 4. *Suppose $\mathfrak{D}$ is an open stability region with piecewise C^1 boundary. Then $\mathfrak{D}$ is a weak Kharitonov region if and only if $\mathfrak{D}$ and its reciprocal*

$$\mathcal{D}^{-1} \doteq \{z : 1/z \in \mathcal{D}\} \tag{15}$$

are both convex.

Turning our attention to the strong problem, we recall that Soh and Foo (1990) studied the case when $\mathcal{D}$ is a special left sector so that the number N_v of vertices is dependent on the angle of the left sector, but is independent on the order of the interval polynomial. We now report their result.

THEOREM 5. *Let the region $\mathcal{D}$ be the left sector whose upper contour is given by*

$$\rho \exp j(a/b)\pi, \tag{16}$$

where $\rho \in [0, \infty)$ and a, b are relatively coprime positive integers satisfying $(1/2) \le (a/b) < 1$. Then

$$N_v = 2b. \tag{17}$$

A Negative Example: The Unit Disk

In the recent literature there are many counterexamples showing that the unit disk is *not* a weak Kharitonov region; e.g., see the paper of Cieslik (1987). However, for a "special interval polynomial" family $\mathcal{P}^0_\infty$, which is a subset of the one defined in (1), Hollot and Bartlett (1988) proved that the unit disk is a weak Kharitonov region.

Formally, for n even, let

$$\begin{aligned}\mathcal{P}^0_\infty = \Big\{&p(s, q) : p(s, q) = \sum_{i=0}^{n-1} q_i s^i + s^n;\\ &q_i^- \le q_i \le q_i^+ \text{ for } i = 0, 1, \ldots, \frac{n}{2};\\ &q_i^- = q_i = q_i^+ \text{ for } i = \frac{n}{2} + 1, \ldots, n - 1\Big\}.\end{aligned} \tag{18}$$

Similarly, for n odd, define

$$\begin{aligned}\mathcal{P}^0_\infty = \Big\{&p(s, q) : p(s, q) = \sum_{i=0}^{n-1} q_i s^i + s^n;\\ &q_i^- \le q_i \le q_i^+ \text{ for } i = 0, 1, \ldots, \frac{n+1}{2};\\ &q_i^- = q_i = q_i^+ \text{ for } i = \frac{n+1}{2} + 1, \ldots, n - 1\Big\}.\end{aligned} \tag{19}$$

THEOREM 6. *The family* $\mathcal{P}_\infty^0$ *defined in* (18) *and* (19) *is Schur if and only if all the vertex polynomials* $p(s, q^v)$ *are Schur.*

Obviously, the number of vertex polynomials is $N_v = \sqrt{2^n}$ for n even and $N_v = \sqrt{2^{n+1}}$ for n odd. For a refinement of Theorem 6, see the work of Barmish (1989).

As a final remark for unit disk stability, we note that the so-called 45° *rotated rectangle* set has been studied by Kraus *et al.* (1988). When the coefficients of a polynomial vary in this uncertainty set, only a subset of the vertex polynomials should be considered; see the work of Mansour and Kraus (1988).

Polytopic Families of Polynomials

In the previous sections, we mainly concentrated on interval, sphere, and diamond of polynomials. Here, we briefly remark that the more general case of a *polytope of polynomials* has been studied and solved by Bartlett *et al.* (1988). The so-called *Edge Theorem* shows that the roots of an n dimensional polytope of polynomials are located on the "one-dimensional exposed edges" of the polytope.

V. COMPUTATIONAL COMPLEXITY

In this section, we evaluate upper bounds (COMP) on the computational complexity of robust stability for some families of polynomials and certain $\mathcal{D}$ regions of interest in many applications.

Left Half Plane and Interval Polynomials

It is well known (see, e.g., the computations reported in the paper of Pace and Barnett, 1973) that the number of multiplications required to check if all roots of a fixed polynomial $p(s, q^*)$ of order n are contained in the open left half plane using the *algorithm of Routh* is

$$\begin{aligned} &\frac{n^2}{4} \qquad n \text{ even;} \\ &\frac{n^2-1}{4} \qquad n \text{ odd.} \end{aligned} \tag{20}$$

We also recall that the number of divisions and additions is equal to the number of multiplications given above. Hence, using the Theorem of Kharitonov, it is straightforward to show that

$$\text{COMP} = O(n^2). \tag{21}$$

More details about alternative algorithms to Routh's can be found in Pace and Barnett (1973) where it is also shown that the computational complexity of the Test of Lienard and Chipart (1914) (see also Volume 2 of Gantmacher, 1960) is $O(n^4)$. This upper bound seems rather weak, since it is well known that the number of determinants involved in the Test of Lienard and Chipart is approximately half of the number of determinants needed in the Test of Routh.

Left Half Plane and Diamond Polynomials

Similarly to the case of interval polynomials, we evaluate the computational complexity for diamond polynomials. In this case, using Theorem 3 and (20), it is straightforward to show that

$$\mathrm{COMP} = O(n^2). \tag{22}$$

Unit Disk and Special Interval Polynomials

The number of elementary operations required to check if all roots of a polynomial $p(s, q^*)$ of order n are contained in the unit disk using the *array of Jury* is given by (see, e.g., Jury, 1964)

$$4n^2 + 8n - 7 \tag{23}$$

multiplications,

$$n^2 + 3n - 1 \tag{24}$$

additions, and two divisions. This computational complexity is lower than the one obtained using the Test of Schur–Cohn (see Pace and Barnett, 1973). Combining the result of Hollot and Bartlett (1986) with (23) and (24), we easily obtain that

$$\mathrm{COMP} = O(2^n \cdot n^2). \tag{25}$$

Class of Left Sectors and Interval Polynomials

Let A^* be the companion matrix associated with the vertex polynomial $p(s, q^*)$. Then the root location sector test given in Davison and Ramesh (1970) (see also Anderson *et al.*, 1974) requires that the eigenvalues of a matrix $\Gamma \in \mathbf{R}^{2n\times 2n}$ defined as

$$\Gamma = \begin{bmatrix} A^* \cos(a/b)\pi & -A^* \sin(a/b)\pi \\ A^* \sin(a/b)\pi & A^* \cos(a/b)\pi \end{bmatrix} \tag{26}$$

lie in the open left half plane. We remark that this eigenvalue problem can be easily reduced from order $2n$ to order n (see p. 46, Vol. 1 of Gantmacher, 1960). Neglecting the cost associated with the computation of the coefficients of the characteristic equation, the eigenvalue problem (26) has computational complexity $O(n^2)$. Hence, invoking (17), we obtain an upper bound on the computational complexity of root location of interval polynomials within the special sector of Soh and Foo (1990),

$$\text{COMP} = O(n^2 \cdot b). \tag{27}$$

VI. Concluding Remarks

We feel that the study of computational complexity for robust stability of uncertain systems is a widely open and promising research area. We conclude this paper by mentioning a class of problems which we think are worth studying. Let $A \in \mathbf{R}^{n\times n}$ be a real matrix whose entries a_{ij} vary between upper and lower bounds

$$a_{ij}^- \leq a_{ij} \leq a_{ij}^+$$

for all $i, j = 1, 2, \ldots, n$. The matrix A is often called an *interval matrix*. The main task is to study robust stability of interval matrices and to evaluate its computational complexity. At present, only negative or partial results have been found; e.g., see the papers of Barmish *et al.* (1988), Barmish and Hollot (1984), and Cobb and Demarco (1989) for specific results and the surveys of Barmish (1988) and Barmish and Kang (to appear) for more general discussions.

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